Dimers on the kagome lattice I: Finite lattices

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Abstract

We report exact results on the enumeration of close-packed dimers on a *finite* kagome lattice with general *asymmetric* dimer weights under periodic and cylindrical boundary conditions. For symmetric dimer weights, the resulting dimer generating functions reduce to very simple expressions, and we show how the simple expressions can be obtained from the consideration of a spin-variable mapping.

Key words: finite kagome lattice, close-packed dimers, spin-variable mapping
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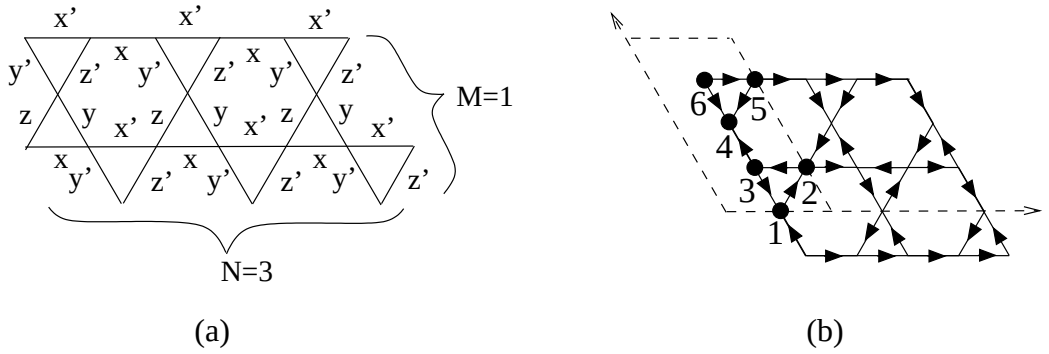
1 Introduction

A central problem in lattice statistics is the enumeration of close-packed dimers on lattices and graphs. The origin of the problem has a long history dating back to a 1937 paper by Fowler and Rushbrooke [1] in an attempt of enumerating the absorption of diatomic molecules on a surface. A breakthrough in dimer statistics has been the exact solution of the generating function for a finite square lattice of size $M \times N$, where M and N are arbitrary, obtained by Kasteleyn [2] and by Temperley and Fisher [3] in 1961.

In view of the role of finite-size solutions in the conformal field theory discovered by Blöte *et al.* [4] in 1986, it has been of increasing importance to consider solutions of lattice models for various finite two-dimensional lattices. Thus, the dimer solution has been extended to cylindrical [5] and nonorientable [6] lattices. However, these lattices are variants of the square lattice which may not necessarily exhibit special lattice-dependence features.



(a) (b)

Fig. 1. (a) An $M \times N$ lattice of $6MN$ sites with asymmetric dimer weights. (b) A Kasteleyn edge orientation of the kagome lattice. A unit cell is the region bounded by broken lines.

In a recent paper [7] we have reported enumeration results of close-packed dimers on an infinite kagome lattice with symmetric dimer weights (activities). The solution turned out to assume a very simple expression. In this paper we extend the solution to *finite* lattices with general *asymmetric* weights and under two different boundary conditions. We find the solutions given by entirely different expressions. For symmetric weights, however, the solutions again reduce to simple expressions. We show how the simple expressions can be deduced quite directly from the consideration of a spin-variable mapping.

2 Finite lattices

Consider a kagome lattice with asymmetric dimer weights x, y, z around up-pointing triangles and x', y', z' around down-pointing triangles as shown in Fig. 1(a). We consider a lattice of $M \times N$ unit cells having a total of $6MN$ lattice sites; the case of $M = 1, N = 3$ is shown in Fig. 1(a). A unit cell of the lattice contains 6 sites numbered $1, 2, ..., 6$ as indicated in Fig. 1(b). The Kasteleyn edge orientations adopted in [7] is also shown in Fig. 1(b).

The method of Pfaffians [2] concerns with the evaluation of a $6MN \times 6MN$ antisymmetric *Kasteleyn matrix* A written down according to edge weights and orientations (under specific boundary conditions) which can be read off from Figs 1(a) and 1(b), and by adopting the prescription

$$A_{ij} = \begin{cases} +w_{ij} \text{ orientation from } i \text{ to } j \\ -w_{ij} \text{ orientation from } j \text{ to } i \end{cases} \tag{1}$$

where w_{ij} is the weight of edge ij. The dimer generating function is then given by the square root of the determinant of the matrix A. We consider the kagome lattice under two different boundary conditions.

2.1 The periodic boundary condition

First we consider the periodic (toroidal) boundary condition (PBC) for which the lattice is periodic in both the horizontal and vertical directions. Kasteleyn [2] has shown that under the PBC the dimer generating function $Z_{\rm PBC}$ is a linear combination of four Pfaffians $\mathrm{Pf}|A_i|, i = 1, 2, 3, 4$,

$$Z_{\rm PBC} = \frac{1}{2}\Big[- \mathrm{Pf}|A_1| + \mathrm{Pf}|A_2| + \mathrm{Pf}|A_3| + \mathrm{Pf}|A_4|\Big]. \tag{2}$$

Up to signs yet to be determined, the Pfaffians are the square root of the determinants specified by the Kasteleyn orientation of lattice edges with, or without, the reversal of arrows on edges connecting two opposite boundaries. A perusal of Fig. 1(b) and the use of the prescription (1) lead to the four $6MN \times 6MN$ Kasteleyn matrices,

$$\begin{aligned}
A_1 =& a_{0,0} \otimes I_M \otimes I_N + a_{1,0} \otimes I_M \otimes T_N - a_{1,0}^T \otimes I_M \otimes T_N^T \\
&+ a_{0,1} \otimes T_M \otimes I_N - a_{0,1}^T \otimes T_M \otimes I_N^T \\
&+ a_{1,1} \otimes T_M \otimes T_N - a_{1,1}^T \otimes T_M^T \otimes T_N^T \\
A_2 =& a_{0,0} \otimes I_M \otimes I_N + a_{1,0} \otimes I_M \otimes H_N - a_{1,0}^T \otimes I_M \otimes H_N^T \\
&+ a_{0,1} \otimes T_M \otimes I_N - a_{0,1}^T \otimes T_M \otimes I_N^T \\
&+ a_{1,1} \otimes T_M \otimes H_N - a_{1,1}^T \otimes T_M^T \otimes H_N^T \\
A_3 =& a_{0,0} \otimes I_M \otimes I_N + a_{1,0} \otimes I_M \otimes T_N - a_{1,0}^T \otimes I_M \otimes T_N^T \\
&+ a_{0,1} \otimes H_M \otimes I_N - a_{0,1}^T \otimes H_M \otimes I_N^T \\
&+ a_{1,1} \otimes H_M \otimes T_N - a_{1,1}^T \otimes H_M^T \otimes T_N^T \\
A_4 =& a_{0,0} \otimes I_M \otimes I_N + a_{1,0} \otimes I_M \otimes H_N - a_{1,0}^T \otimes I_M \otimes H_N^T \\
&+ a_{0,1} \otimes H_M \otimes I_N - a_{0,1}^T \otimes H_M \otimes I_N^T \\
&+ a_{1,1} \otimes H_M \otimes H_N - a_{1,1}^T \otimes H_M^T \otimes H_N^T.
\end{aligned} \tag{3}$$

Here, the superscripts T denote transpose, $\otimes$ is direct product, I_M is the $M \times M$ identity matrix, and H_N, T_N are the $N \times N$ matrices

$$H_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad T_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad F_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{4}$$

where F_N is to be used later in (15), and

$$a_{0,0} = \begin{pmatrix} 0 & z' & -y' & 0 & 0 & 0 \\ -z' & 0 & x' & 0 & 0 & 0 \\ y & -x' & 0 & y & 0 & 0 \\ 0 & 0 & -y & 0 & -z' & -y' \\ 0 & 0 & 0 & z' & 0 & -x' \\ 0 & 0 & 0 & y' & x' & 0 \end{pmatrix}, \quad a_{1,0} = \begin{pmatrix} 0\,0\,0 & 0 & 0\,0 \\ 0\,0\,x & -z & 0\,0 \\ 0\,0\,0 & 0 & 0\,0 \\ 0\,0\,0 & 0 & 0\,0 \\ 0\,0\,0 & 0 & 0\,x \\ 0\,0\,0 & 0 & 0\,0 \end{pmatrix},$$

$$a_{0,1} = \begin{pmatrix} 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \\ y\,0\,0\,0\,0\,0 \end{pmatrix}, \quad a_{1,1} = \begin{pmatrix} 0 & 0\,0\,0\,0\,0 \\ 0 & 0\,0\,0\,0\,0 \\ 0 & 0\,0\,0\,0\,0 \\ 0 & 0\,0\,0\,0\,0 \\ -z & 0\,0\,0\,0\,0 \\ 0 & 0\,0\,0\,0\,0 \end{pmatrix}, \tag{5}$$

$$a_{-1,0} = -a_{1,0}^T, \quad a_{0,-1} = -a_{0,1}^T, \quad a_{-1,-1} = -a_{1,1}^T.$$

The determinant of a matrix is equal to the product of its eigenvalues. To determine eigenvalues of $A_{1,2,3,4}$, we first block-diagonalize the 4 matrices by appropriate Fourier transforms. Since T_N and T_N^T commute, they can be simultaneously diagonalized and replaced by respective eigenvalues $e^{i\theta_n}$ and $e^{-i\theta_n}$, where $\theta_n = 2\pi n/N,\ n = 0, \ldots, N-1$.

Similarly, T_M and T_M^T can be simultaneously diagonalized and replaced by eigenvalues $e^{i\phi_m}$ and $e^{-i\phi_m}$, where $\phi_m = 2\pi m/M,\ m = 0, \ldots, M-1$.

Likewise, H_N and H_N^T commute and they can be simultaneously diagonalized and replaced by eigenvalues $e^{i\alpha_n}$ and $e^{-i\alpha_n}$, where $\alpha_n = (2n+1)\pi/N,\ n = 0, \ldots, N-1$; H_M and H_M^T can be simultaneously diagonalized and replaced by eigenvalues $e^{i\beta_m}$ and $e^{-i\beta_m}$, where $\beta_m = (2m+1)\pi/M,\ m = 0, \ldots, M-1$. Then we find

$$\begin{aligned} \det|A_1| &= \prod_{n=0}^{N-1}\prod_{m=0}^{M-1} \det|A(\theta_n, \phi_m)|, & \det|A_2| &= \prod_{n=0}^{N-1}\prod_{m=0}^{M-1} \det|A(\alpha_n, \phi_m)|, \\ \det|A_3| &= \prod_{n=0}^{N-1}\prod_{m=0}^{M-1} \det|A(\theta_n, \beta_m)|, & \det|A_4| &= \prod_{n=0}^{N-1}\prod_{m=0}^{M-1} \det|A(\alpha_n, \beta_m)|, \end{aligned} \tag{6}$$

where the 6×6 matrix $A(\theta, \phi)$ is anti-hermitian and is given by

$$
\begin{aligned}
A(\theta,\phi) = {} & a_{0,0} + a_{1,0}e^{i\theta} + a_{-1,0}e^{-i\theta} + a_{0,1}e^{i\phi} + a_{0,-1}e^{-i\phi} + a_{1,1}e^{i(\theta+\phi)} \\
& + a_{-1,-1}e^{-i(\theta+\phi)} \\
= {} & \begin{pmatrix}
0 & z' & -y' & 0 & ze^{-i(\theta+\phi)} & -ye^{-i\phi} \\
-z' & 0 & x'+xe^{i\theta} & -ze^{i\theta} & 0 & 0 \\
y & -x'-xe^{-i\theta} & 0 & y & 0 & 0 \\
0 & ze^{-i\theta} & -y & 0 & -z' & -y' \\
-ze^{i(\theta+\phi)} & 0 & 0 & z' & 0 & -x'+xe^{i\theta} \\
ye^{i\phi} & 0 & 0 & y' & x'-xe^{-i\theta} & 0
\end{pmatrix}
\end{aligned}
\tag{7}
$$

This yields

$$
\det A(\theta,\phi) = 2A + 2D\cos(\phi) + 2E\cos(2\theta+\phi) + 4\Delta_2 \sin^2\theta
$$

with

$$
\begin{aligned}
A &= (xy'z + x'yz')^2 + (x'y'z + xyz')^2 \\
D &= -(xy'z - x'yz')^2 \\
E &= (x'y'z - xyz')^2 \\
\Delta_2 &= (x'yz - xy'z')^2.
\end{aligned}
\tag{8}
$$

The desired generating function is now obtained by substituting either $\mathrm{Pf}|A_i| = +\sqrt{\det|A_i|}$ or $-\sqrt{\det|A_i|}$ into (2). In the present case the signs in front of the square roots can be determined by considering the case of $M = N = 1$. By explicit enumeration we have

$$
Z_{\mathrm{PBC}} = 2(x+x')(yz'+y'z), \qquad M = N = 1. \tag{9}
$$

It is readily verified that the expression (9) is reproduced by (2) if all 4 terms in (2) are positive. Thus, we are led to the final expression

$$
Z_{\mathrm{PBC}} = \frac{1}{2}\Big[\sqrt{\det|A_1|} + \sqrt{\det|A_2|} + \sqrt{\det|A_3|} + \sqrt{\det|A_4|}\Big], \tag{10}
$$

where $\det|A_i|$, $i = 1, ..., 4$, are given by (6).

For symmetric dimer weights $x' = x, y' = y, z' = z$, we have $D = E = \Delta_2 = 0$, $\det|A_{1,2,3,4}| = (4xyz)^{MN}$, and the simple result

$$
Z_{\mathrm{PBC}} = 2 \cdot (4xyz)^{MN}, \quad x' = x,\ y' = y,\ z' = z. \tag{11}
$$

We shall see in Sec. 3 that this simple result can be understood and deduced directly using a spin-variable mapping.

In the case of an infinite lattice, (2) leads to the per-dimer free energy

$$\begin{aligned} f &= \lim_{M,N\to\infty} \frac{1}{3MN} \ln Z_{\rm PBC} \\ &= \frac{1}{24\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln \Big[2A + 2D\cos(\theta-\phi) \\ &\qquad +2E\cos(\theta+\phi) + 4\Delta_2 \sin^2\theta \Big]. \end{aligned} \tag{12}$$

This free energy is independent of the boundary condition. The free energy (12) can also be deduced using the vertex-model approach introduced in [7], details of which are straightforward and will not be given.

For symmetric dimer weights $x' = x, y' = y, z' = z$, (12) reduces further to

$$f = \frac{1}{3} \ln(4xyz). \tag{13}$$

This result for an infinite lattice was first reported in [8] with the full derivation given in [7]. The exact per-dimer entropy $s = \frac{2}{3}\ln 2$ obtained from (13) at $x = y = z = 1$ has been cited earlier by Phares and Wunderlich from [9] and by Elser [10] from different considerations.

2.2 The cylindrical boundary condition

Consider next the cylindrical boundary condition (CBC) for which the lattice of $M \times N$ unit cells is periodic in the horizontal direction. The Kasteleyn orientation is achieved by reversing the orientations of the $4M-1$ edges connecting unit cells in the Nth column to those in the first column. This gives the dimer generating function as a single Pfaffian

$$Z_{\rm CBC} = \sqrt{\det |A_{\rm CBC}|}, \tag{14}$$

where $A_{\rm CBC}$ is the $6MN \times 6MN$ matrix

$$\begin{aligned} A_{\rm CBC} =& a_{0,0} \otimes I_M \otimes I_N + a_{1,0} \otimes I_M \otimes H_N - a_{1,0}^T \otimes I_M \otimes H_N^T + a_{0,1} \otimes F_M \otimes I_N \\ & - a_{0,1}^T \otimes F_M \otimes I_N^T + a_{1,1} \otimes F_M \otimes H_N - a_{1,1}^T \otimes F_M^T \otimes H_N^T. \end{aligned} \tag{15}$$

Here, matrices a are those in (5) and F_N has been given in (4). Again, the matrix $A_{\rm CBC}$ is block-diagonalized by replacing H_N and H_N^T by their respective eigenvalues. This leads to

$$\det |A_{\rm CBC}| = \prod_{n=0}^{N-1} \det |B_M(\theta_n)|, \qquad \theta_n = (2n+1)\pi/N \tag{16}$$

where $B_M(\theta)$ is the $6M \times 6M$ Kasteleyn matrix

$$
\begin{aligned}
B_M(\theta) &= B \otimes I_M + B_+ \otimes F_M + B_- \otimes F_M^T \\
&= \begin{pmatrix}
B & B_+ & 0 & \cdots & 0 & 0 & 0 \\
B_- & B & B_+ & \cdots & 0 & 0 & 0 \\
0 & B_- & B & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & B & B_+ & 0 \\
0 & 0 & 0 & \cdots & B_- & B & B_+ \\
0 & 0 & 0 & \cdots & 0 & B_- & B
\end{pmatrix},
\end{aligned}
\tag{17}
$$

$B = a_{0,0} + e^{i\theta} a_{1,0} - e^{-i\theta} a_{1,0}^T$, $B_+ = a_{0,1} + e^{i\theta} a_{1,1}$, and $B_- = -a_{0,1}^T - e^{-i\theta} a_{1,1}^T$ are the 6×6 matrices

$$
B = \begin{pmatrix}
0 & z' & -y' & 0 & 0 & 0 \\
-z' & 0 & x' + xe^{i\theta} & -ze^{i\theta} & 0 & 0 \\
y' & -x' - xe^{-i\theta} & 0 & y & 0 & 0 \\
0 & ze^{-i\theta} & -y & 0 & -z' & -y' \\
0 & 0 & 0 & z' & 0 & -x' + xe^{i\theta} \\
0 & 0 & 0 & y' & x' - xe^{-i\theta} & 0
\end{pmatrix},
\tag{18}
$$

$$
B_+ = \begin{pmatrix}
0 & 0\,0\,0\,0\,0 \\
0 & 0\,0\,0\,0\,0 \\
0 & 0\,0\,0\,0\,0 \\
0 & 0\,0\,0\,0\,0 \\
-ze^{i\theta} & 0\,0\,0\,0\,0 \\
y & 0\,0\,0\,0\,0
\end{pmatrix},
\qquad
B_- = \begin{pmatrix}
0\,0\,0\,0 & ze^{-i\theta} & -y \\
0\,0\,0\,0 & 0 & 0 \\
0\,0\,0\,0 & 0 & 0 \\
0\,0\,0\,0 & 0 & 0 \\
0\,0\,0\,0 & 0 & 0
\end{pmatrix}.
$$

The matrix $B_M(\theta)$ is of a form of that occurring in the evaluation of an Ising partition function under the cylindrical boundary condition [5,6], and the determinant $\det|B_M(\theta)|$ can be evaluated as follows:

Let $B^{i,j}$ denote the 6×6 matrix B with row i and column j removed. By Laplacian expansion and the use of a lemma established in [6], the determinant of B_M can be expanded as

$$\begin{aligned}\det|B_M| =& \det|B|\cdot\det|B_{M-1}| + z^2\det|B^{5,5}|\cdot\det|B^{1,1}_{M-1}| \\ &+y^2\det|B^{6,6}|\cdot\det|B^{1,1}_{M-1}| + yze^{i\theta}\det|B^{5,6}|\cdot\det|B^{1,1}_{M-1}| \\ &+yze^{-i\theta}\det|B^{6,5}|\cdot\det|B^{1,1}_{M-1}|. \end{aligned} \tag{19}$$

Similarly, the determinant of the matrix $B^{1,1}_M$ can be expanded as

$$\begin{aligned}\det|B^{1,1}_M| =& \det|B^{1,1}|\cdot\det|B_{M-1}| + z^2\det|B^{1,1;5,5}|\cdot\det|B^{1,1}_{M-1}| \\ &+y^2\det|B^{1,1;6,6}|\cdot\det|B^{1,1}_{M-1}| + yze^{i\theta}\det|B^{1,1;5,6}|\cdot\det|B^{1,1}_{M-1}| \\ &+yze^{-i\theta}\det|B^{1,1;6,5}|\cdot\det|B^{1,1}_{M-1}|, \end{aligned} \tag{20}$$

where $B^{i,j;k,\ell}$ is the 6×6 matrix B with rows i and k, and columns j and ℓ deleted.

Write $\mathcal{B}_M \equiv \det|B_M|$ and $\mathcal{C}_M \equiv \det|B^{1,1}_M|$. Expansions (19) and (20) are recursion relations of $\mathcal{B}$ and $\mathcal{C}$,

$$\begin{aligned}\mathcal{B}_M &= a\,\mathcal{B}_{M-1} + b\,\mathcal{C}_{M-1} \\ \mathcal{C}_M &= c\,\mathcal{B}_{M-1} + d\,\mathcal{C}_{M-1}\end{aligned} \tag{21}$$

where

$$\begin{aligned}a =& \det|B| \\ b =& z^2\det|B^{5,5}| + y^2\det|B^{6,6}| + yze^{i\theta}\det|B^{5,6}| + yze^{-i\theta}\det|B^{6,5}| \\ c =& \det|B^{1,1}| \\ d =& z^2\det|B^{1,1;5,5}| + y^2\det|B^{1,1;6,6}| + yze^{i\theta}\det|B^{1,1;5,6}| + yze^{-i\theta}\det|B^{1,1;6,5}|,\end{aligned} \tag{22}$$

subject to the initial condition $\mathcal{B}_0 = 1$, $\mathcal{C}_0 = 0$. Explicitly using (18), (22) reads

$$\begin{aligned}a =& (x^2+x'^2)(y'^2z^2+y^2z'^2) + 4x^2y'^2z'^2 - 2(x'y'z+xyz')(xy'z+x'yz')\cos\theta \\ &+ 4xy'z'(x'yz-xy'z')\cos^2\theta \\ b =& 2i(x'yz+xy'z')(y'^2z^2+y^2z'^2)\sin\theta - 4iyy'zz'(x'yz-xy'z')\cos\theta\sin\theta \\ c =& -2i\Big[(x^2+x'^2)(x'yz+xy'z') - 2xx'(x'yz-xy'z')\cos\theta\Big]\sin\theta \\ d =& (x^2+x'^2)(y'^2z^2+y^2z'^2) + 4x'^2y^2z^2 + 2(x'y'z+xyz')(xy'z+x'yz')\cos\theta \\ &- 4x'yz(x'yz-xy'z')\cos^2\theta.\end{aligned} \tag{23}$$

The recursion relation (21) can be solved by introducing generating functions

$$\mathbf{B}(t) = \sum_{M=0}^{\infty}\mathcal{B}_M t^M, \quad \mathbf{C}(t) = \sum_{M=0}^{\infty}\mathcal{C}_M t^M. \tag{24}$$

The recursion relation (21) gives

$$\begin{aligned}\mathbf{B}(t) &= 1 + t\,[a\,\mathbf{B}(t) + b\,\mathbf{C}(t)] \\ \mathbf{C}(t) &= t\,[c\,\mathbf{B}(t) + d\,\mathbf{C}(t)],\end{aligned} \tag{25}$$

where we have made use of the initial condition $\mathcal{B}_0 = 1$, $\mathcal{C}_0 = 0$.

Solving (25) for $\mathbf{B}(t)$ and $\mathbf{C}(t)$, we obtain

$$\mathbf{B}(t) = \frac{1 - d\,t}{1 - (a+d)t + (a\,d - b\,c)t^2} = \frac{1 - d\,t}{(1 - \lambda_+ t)(1 - \lambda_- t)}, \tag{26}$$

where

$$\begin{aligned}\lambda_\pm =& (a+d)/2 \pm \sqrt{(a-d)^2/4 + bc} \\ =& A + 2\Delta_2 \sin^2\theta \pm \sqrt{(A + 2\Delta_2 \sin^2\theta)^2 - D^2 - E^2 - 2DE\cos^2\theta}.\end{aligned} \tag{27}$$

Here, A, D, E, Δ_2 have been given in (8). Partial fraction and expand the right-hand side of (26), and compare the resulting expansion with (24), one obtains

$$\mathcal{B}_M(\theta) = \frac{\lambda_+^{M+1} - \lambda_-^{M+1}}{\lambda_+ - \lambda_-} - \frac{\lambda_+^M - \lambda_-^M}{\lambda_+ - \lambda_-} \cdot d. \tag{28}$$

Finally, by combining (14) and (16), we obtain the desired generating function

$$Z_{\rm CBC} = \sqrt{\prod_{n=0}^{N-1} \mathcal{B}_M(\theta_n)}, \qquad \theta_n = (2n+1)\pi/N.$$

For symmetric weights $x' = x, y' = y, z' = z$ we have

$$\begin{aligned} a &= 8x^2y^2z^2(1 - \cos\theta), \quad & b &= 8ixy^3z^3\sin\theta \\ c &= -8ix^3yz\sin\theta, & d &= 8x^2y^2z^2(1 + \cos\theta) \\ \lambda_+ &= 16x^2y^2z^2, & \lambda_- &= 0. \end{aligned}$$

We obtain $\mathcal{B}_M(\theta) = (16x^2y^2z^2)^M \sin^2(\theta/2)$ and hence

$$Z_{\rm CBC} = (4xyz)^{MN} \prod_{n=0}^{N-1} \sin(\theta_n/2) = 2^{1-N} \cdot (4xyz)^{MN}. \tag{29}$$

Again, we shall see in Sec. 3 that this simple expression can be understood using a spin-variable mapping.

In the case of an infinite lattice and since $\lambda_+ > \lambda_-$, we have from (28) $\mathcal{B}_M(\theta) \to \lambda_+^M$. This gives the per-dimer free energy

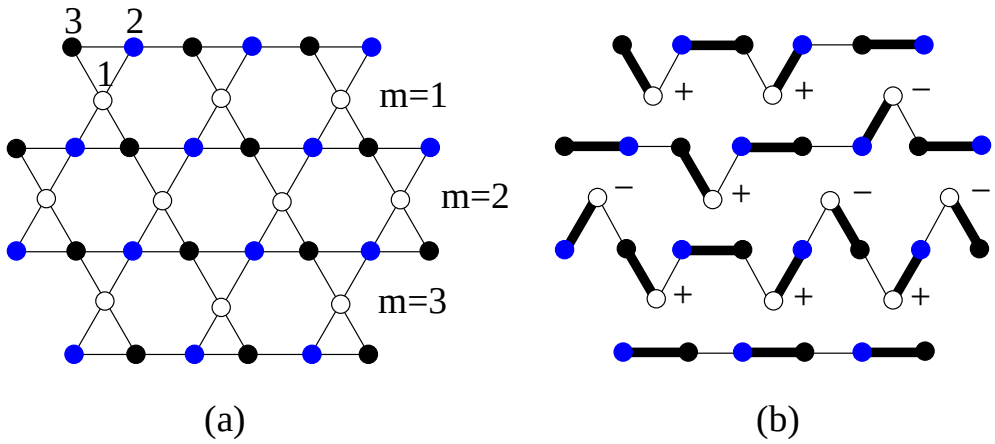


Fig. 2. (a) An $\mathcal{M} = 3$ kagome lattice. Sublattice 1 sites are denoted by open circles. (b) The spin configuration deduced from a typical dimer configuration. The lattice is decomposed into 4 strips or loops depending on the boundary condition (see text).

$$f = \lim_{M,N\to\infty} \frac{1}{3MN} \ln Z_{CBC} = \lim_{N\to\infty} \frac{1}{6N} \sum_{n=0}^{N-1} \ln \lambda_+(\theta_n)$$
$$= \frac{1}{12\pi} \int_0^{2\pi} \ln \lambda_+(\theta)\, d\theta \tag{30}$$

where $\lambda_+(\theta)$ is given by (27). It is readily verified that (30) is identical to the free energy (12) after carrying out the integration over ϕ in (12).

3 A spin-variable mapping

The exact enumeration results (11) and (29) for a lattice of $M \times N$ unit cells with symmetric dimer weights are strikingly simple, suggesting the possibility of a simple derivation. Indeed, Zeng and Elser [11] and Misguich *et al.* [12] have introduced a pseudo-spin consideration of enumerating quantum states which can be transcribed to the present $x = y = z = 1$ case [13]. However, the pseudo-spin consideration was presented in contexts of spin 1/2 antiferromagnets and quantum dimer models, and the application to the classical dimer problem with general weights x, y, z is not immediately obvious. A simpler formulation is very much needed.

We elucidate the matter by describing an alternate spin-variable mapping valid for general x, y, z. First we note that the numbers of x, y, and z dimers are always fixed for finite lattices. This is due to the fact that the three principal axes do not intersect at common points. The kagome lattice has three sublattices as numbered in Fig. 2(a). Denote the number of sites on sublattice i by N_i, $i = 1, 2, 3$, and the number of x dimers by N_x, etc. Then as a consequence of the fact the principal axes do not intersect at common points, we have the

relations

$$N_y + N_z = N_1, \quad N_z + N_x = N_2, \quad N_x + N_y = N_3\,.$$

This gives rise to $N_x = (N_2 + N_3 - N_1)/2$, etc., which are fixed numbers. The dimer generating function is therefore a single monomial of the form

$$Z = \Omega\, x^{N_x} y^{N_y} z^{N_z},$$

so we need only to compute the constant Ω. In the case of $N_1 = N_2 = N_3 = \mathcal{N}$ we are considering, this leads to the expression

$$Z = \Omega\, (xyz)^{\mathcal{N}/2}, \tag{31}$$

where we have $\mathcal{N} = 2MN$ for both the PBC and CBC boundary conditions.

We next map dimer configurations on the lattice to spin configurations on one sublattice, say, 1. Consider the example of the lattice shown in Fig. 2(a). Let the lattice consist $\mathcal{M}$ rows of sublattice 1 sites and $\mathcal{M}+1$ rows of equal 2 and 3 sites with open boundaries in the vertical direction. The boundary condition in the horizontal direction can be either open or periodic. Denote the number of sublattice 1 sites in the mth row by n_m, which must satisfy the sum rule

$$n_1 + n_2 + \cdots + n_{\mathcal{M}} = N_1 = \text{even}, \tag{32}$$

since the lattice must admit dimer coverings.

Assign spin variable

$$\sigma_{mn} = \pm 1, \qquad m = 1, ..., \mathcal{M},\ n = 1, ..., n_m$$

to sublattice 1 sites, where m is the row number counting beginning from the top. Adopt the convention that $\sigma = +1$ (-1) if the dimer covering the site also covers a site above (below) the row. For $+1$ (-1) sites we remove the two edges below (above) the site as well as the horizontal edge directly above (below) it as shown in Fig. 2(b). This procedure decomposes the lattice into strips (loops) for open (periodic) boundary conditions in the horizontal direction.

Now every strip (or loop) must have an even number of sites to accommodate one (or 2) dimer covering(s). This condition imposes constraints on spin configurations that can be realized by this mapping. To best describe the constraints it is convenient to define a row variable

$$\tau_m = \prod_{n=1}^{n_m} \sigma_{mn}, \qquad m = 1, 2, ..., \mathcal{M}. \tag{33}$$

It is then readily verified that we must have

$$
\begin{aligned}
&\tau_1 = (-1)^{n_1}, \\
&\tau_{m-1}\tau_m = (-1)^{n_m}, \qquad m = 2, 3, ..., \mathcal{M}\,, \\
&\tau_{\mathcal{M}} = 1.
\end{aligned} \tag{34}
$$

The constraint $\tau_{\mathcal{M}} = 1$ is automatically satisfied due to (32) and the fact that

$$
\tau_M = (\tau_1)(\tau_1\tau_2)\cdots(\tau_{M-1}\tau_M) = (-1)^{m_1 + \cdots + m_M} = (-1)^{N_1} = 1. \tag{35}
$$

We remark that the row variable (33) can also be used to analyze the dimer models in higher dimensions considered in [14].

We can now compute the constant Ω. Beginning with an overall spin state degeneracy 2^{N_1} of sublattice 1 sites, each constraint in (34) reduces the spin states by a factor of 2. Since there are $\mathcal{M} - 1$ such constraints, we have

$$
\Omega = \begin{cases} 2^{N_1 - (\mathcal{M}-1)}, & \text{open boundaries} \\ 2^{N_1+2}, & \text{horizontal periodic boundary condition} \end{cases} \tag{36}
$$

Note that there is an extra factor $2^{\mathcal{M}+1}$ for periodic boundary conditions in the horizontal direction since each loop has 2 dimer coverings. Expression (36) is a very general result independent of specific values of n_m.

For a lattice of $M \times N$ unit cells with toroidal boundary conditions PBC considered in Sec. 2.1, we have $N_1 = N_2 = N_3 = \mathcal{N} = 2MN$, $\mathcal{M} = 2M$. Hence (36) gives $\Omega = 2^{N_1} \cdot 2^{-(\mathcal{M}-1)} \cdot 2^{\mathcal{M}}$, where as in (36) the second factor is due to $\mathcal{M} - 1$ constraints with the $\mathcal{M}$th constraint automatically satisfied, and the third factor is due to the 2-fold dimer coverings of each loop. This leads to $Z_{\rm PBC} = 2 \cdot (4xyz)^{MN}$ in agreement with (11).

For a lattice of $M \times N$ unit cells with cylindrical boundary condition CBC considered in Sec. 2.2, we again have $N_1 = N_2 = N_3 = \mathcal{N} = 2MN$, $\mathcal{M} = 2M$. However, the $\mathcal{M}$th row of N sublattice 1 spins must be all $+1$ (Cf. Fig. 2(b) with the bottom row of sites removed) reducing the counting by a factor of 2^{-N} and the number of rows by 1. Hence $\Omega = 2^{-N} \cdot 2^{N_1 - (\mathcal{M}-1)} \cdot 2^{\mathcal{M}}$ and $Z_{\rm CBC} = 2^{1-N} \cdot (4xyz)^{MN}$ in agreement with (29).

We remark that our spin-variable mapping is akin to one used recently by Dhar and Chandra [14]. However, the Dhar-Chandra approach focuses on an infinite lattice by ignoring what happens on the boundary. Here, we treat the boundary effect rigorously and apply the mapping to finite lattices.

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References

[1] R. H. Fowler and G. S. Rushbrooke, *Trans. Faraday Soc.*, **33** (1937), p.1272

[2] P. W. Kasteleyn, *Physica (Amsterdam)*, **27** (1961), p.1209

[3] H. N. V. Temperley and M. E. Fisher, *Phil. Mag.*, **6** (1961), p.1061 ; M. E. Fisher, *Phys. Rev.*, **124** (1961), p.1664

[4] H. W. J. Blóte, J. Cardy, M. P. Nightingale, *Phys. Rev. Lett.*, **56** (1986), p.742

[5] B. McCoy and T. T. Wu, *The two dimensional Ising model*, Harvard University Press, 1973

[6] W. T. Lu and F. Y. Wu, *Phys. Lett. A*, **258** (1998), p.157 ; *ibid.*, **293** (2002), p.235

[7] F. Wang and F. Y. Wu, *Phys. Rev. E*, **75** (2007), p.040105(R)

[8] F. Y. Wu, *Int. J. Mod. Phys. B*, **20** (2006), p.5357

[9] A. J. Phares and F. J. Wunderlich, *Nuovo Cimento Soc. Ital. Fis. B*, **101** (1988), p.653

[10] V. Elser, *Phys. Rev. Lett.*, **62** (1989), p.2405

[11] C. Zeng and V. Elser, *Phys. Rev. B*, **51** (1995), p.8318

[12] G. Misguich, D. Serban and V. Pasquier, *Phys. Rev. B*, **67** (2003), p.214413

[13] G. Misguich, private communication.

[14] D. Dhar and S. Chandra, arXiv:0711.0971

Dimers on the kagome lattice II: Correlations and the Grassmannian approach

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Abstract

In this paper we continue our consideration of closed-packed dimers on the kagome lattice. Using the Pfaffian approach we evaluate the correlation between dimers on two lattice edges. It is found that the correlation is extremely short-ranged in the case of symmetric dimers weights. Explicit expressions for the nonvanishing correlations are obtained in the interior of a large lattice. We also describe a Grassmannian functional integral approach, and use it to evaluate the dimer generating function and correlation functions.

Key words: kagome lattice, close-packed dimers, correlation function, Grassmannian approach
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1 Introduction

in the preceeding paper [1], hereafter referred to as I, we presented exact results on the generating function for closed-packed dimers on a finite kagome lattice with asymmetric dimer weights. To further illustrate the usefulness of the Pfaffian method used in I, in this paper we extend the consideration to dimer-dimer correlations. For symmetric dimer weights we find that the correlation is extremely short-ranged, a property unique to the kagome lattice and previously reported by us in [2]. Here we derive explicit expressions of nonvanishing correlation functions for a large lattice. We also describe the formulation of a Grassmannian function integral approach, and use it to evaluate the dimer generating funciton and correlation functions.

2 The dimer-dimer correlation function

The dimer-dimer correlation function measures the correlation between dimers on two lattice edges. This correlation is best described by introducing an edge occupation number

$$n_{ij} = \begin{cases} 1 \text{ if edge } ij \text{ is occupied by a dimer} \\ 0 \text{ if edge } ij \text{ is empty.} \end{cases} \tag{1}$$

Likewise, the edge vacancy number is $\bar{n}_{ij} = 1 - n_{ij}$. The correlation function between two dimers covering edge ij in unit cell at $\mathbf{r}_1$ and edge $k\ell$ in unit cell at $\mathbf{r}_2$ is defined by

$$\begin{aligned} c(ij, \mathbf{r}_1; k\ell, \mathbf{r}_2) &= \langle n_{ij,\mathbf{r}_1}\, n_{k\ell,\mathbf{r}_2} \rangle - \langle n_{ij,\mathbf{r}_1} \rangle \langle n_{k\ell,\mathbf{r}_2} \rangle \\ &= \langle \bar{n}_{ij,\mathbf{r}_1}\, \bar{n}_{k\ell,\mathbf{r}_2} \rangle - \langle \bar{n}_{ij,\mathbf{r}_1} \rangle \langle \bar{n}_{k\ell,\mathbf{r}_2} \rangle \end{aligned} \tag{2}$$

where $\langle \cdot \rangle$ denotes the configuration average.

The second line in (2) is useful in computing the correlation function in the Pfaffian approach [3], since using it we need only to keep track of the dimer generating function with specific edge(s) missing as dictated by $\langle \bar{n} \rangle$ or $\langle \bar{n}\bar{n} \rangle$.

Let A be the antisymmetric Kasteleyn matrix derived from a Kasteleyn orientation, and let A' denote the antisymmetric matrix derived from A with edge ij, say in computing $\langle \bar{n}_{ij} \rangle$, missing. Write

$$Z = \mathrm{Pf} A, \quad Z' = \mathrm{Pf} A' = \mathrm{Pf}[A + \Delta] \tag{3}$$

where Δ is the matrix with zero elements everywhere except the ij element is $-A_{ij}$ and the ji element is $-A_{ji}$. Then

$$\langle \bar{n}_{ij} \rangle = Z'/Z = \mathrm{Pf} A'/\mathrm{Pf} A$$

and

$$\langle \bar{n}_{ij} \rangle^2 = \frac{\det A'}{\det A} = \frac{\det[A(I + G\Delta)]}{\det A} = \det(I + G\Delta). \tag{4}$$

where $G \equiv A^{-1}$ is the Green's function matrix and I the identity matrix.

In (4) we need only to keep those row(s) and column(s) in Δ and A^{-1} where elements of Δ are nonzero. This effectively reduces dimensions of matrices G and Δ to at most 4×4 in the computation of (2). Explicitly, we have

$$\langle \bar{n}_{ij,\mathbf{r}_1} \rangle^2 = \det \left| I_2 + \begin{pmatrix} G(\mathbf{r}_1,\mathbf{r}_2)_{ii} & G(\mathbf{r}_1,\mathbf{r}_2)_{ij} \\ G(\mathbf{r}_1,\mathbf{r}_2)_{ji} & G(\mathbf{r}_1,\mathbf{r}_2)_{jj} \end{pmatrix} \begin{pmatrix} 0 & -A_{ij} \\ -A_{ji} & 0 \end{pmatrix} \right|$$

$$\langle \bar{n}_{ij,\mathbf{r}_1}\, \bar{n}_{k\ell,\mathbf{r}_2} \rangle^2 = \det |I_4 + \tilde{G}\tilde{\Delta}|, \tag{5}$$

where I_n is the $n \times n$ identity matrix, $\tilde{G}$ and $\tilde{\Delta}$ are the 4×4 matrices

$$\tilde{G} = \begin{pmatrix} G(\mathbf{r}_1,\mathbf{r}_1)_{ii} & G(\mathbf{r}_1,\mathbf{r}_1)_{ij} & G(\mathbf{r}_1,\mathbf{r}_2)_{ik} & G(\mathbf{r}_1,\mathbf{r}_2)_{i\ell} \\ G(\mathbf{r}_1,\mathbf{r}_1)_{ji} & G(\mathbf{r}_1,\mathbf{r}_1)_{jj} & G(\mathbf{r}_1,\mathbf{r}_2)_{jk} & G(\mathbf{r}_1,\mathbf{r}_2)_{j\ell} \\ G(\mathbf{r}_2,\mathbf{r}_1)_{ki} & G(\mathbf{r}_2,\mathbf{r}_1)_{kj} & G(\mathbf{r}_2,\mathbf{r}_2)_{kk} & G(\mathbf{r}_2,\mathbf{r}_2)_{k\ell} \\ G(\mathbf{r}_2,\mathbf{r}_1)_{\ell i} & G(\mathbf{r}_2,\mathbf{r}_1)_{\ell j} & G(\mathbf{r}_2,\mathbf{r}_2)_{\ell k} & G(\mathbf{r}_2,\mathbf{r}_2)_{\ell\ell} \end{pmatrix},$$

$$\tilde{\Delta} = \begin{pmatrix} 0 & -A_{ij} & 0 & 0 \\ -A_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_{k\ell} \\ 0 & 0 & -A_{\ell k} & 0 \end{pmatrix}.$$

The formulation so far is very general applicable to any finite lattice with symmetric or asymmetric dimer weights and $\mathbf{r}_1$ and $\mathbf{r}_2$ arbitrary. We now specialize to a large lattice with symmetric dimer weights.

In the interior of a large lattice, the correlation depends only on the difference $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = \{r_x, r_y\}$, so elements of G are given by

$$G(\{r_{1x}, r_{1y}\}; \{r_{2x}, r_{2y}\})_{ij} = \int_0^{2\pi} \int_0^{2\pi} \frac{d\theta\, d\phi}{(2\pi)^2}\, e^{i[(r_{1x}-r_{2x})\theta + (r_{1y}-r_{2y})\phi]} A^{-1}(\theta,\phi)_{ij}. \tag{6}$$

For symmetric weights, the 6×6 inverse matrix $A^{-1}(\theta,\phi)$ in (6) is computed by using Equ. (7) in I, yielding

$$A^{-1}(\theta,\phi) = \frac{1}{4xyz} \begin{pmatrix} P_{3\times3} + Q_{3\times3} & R_{3\times3} \\ -R^{\dagger}_{3\times3} & P_{3\times3} - Q_{3\times3} \end{pmatrix}, \tag{7}$$

where

$$P_{3\times3} = \begin{pmatrix} 0 & xye^{-i\theta} & xz \\ -yxe^{i\theta} & 0 & -yze^{i\theta} \\ -zx & zye^{-i\theta} & 0 \end{pmatrix}, \quad Q_{3\times3} = \begin{pmatrix} x^2(e^{-i\theta} - e^{i\theta}) & -xy & -xze^{i\theta} \\ yx & 0 & -yz \\ zxe^{-i\theta} & zy & 0 \end{pmatrix},$$

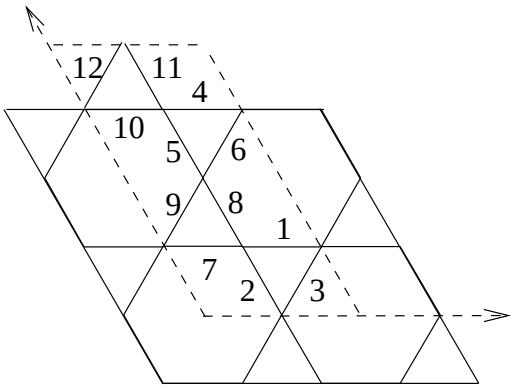

Fig. 1. Labeling of the twelve edges belonging to one unit cell.

$$R_{3\times 3} = \begin{pmatrix} 0 & -xy(e^{-i(\theta+\phi)} + e^{-i\phi}) & xz(e^{-i(\theta+\phi)} + e^{-i\phi}) \\ yx(e^{i\theta} - 1) & y^2(1 - e^{-i\phi}) & yz(e^{-i\phi} + e^{i\theta}) \\ zx(e^{i\theta} - 1) & zy(1 - e^{-i(\theta+\phi)}) & z^2(e^{-i(\theta+\phi)} + e^{i\theta}) \end{pmatrix}$$

and the superscript † denotes Hermitian conjugation.

Substituting (7) into (6), we find $G(\{r_{1x}, r_{1y}\}; \{r_{2x}, r_{2y}\})_{ij} = 0$ if $|r_{1x} - r_{2x}| \geq 2$ or $|r_{1y} - r_{2y}| \geq 2$, or, explicitly,

$$c(ij, \mathbf{r}_1; k\ell, \mathbf{r}_2) = 0, \quad |\mathbf{r}_1 - \mathbf{r}_2| \geq 2. \tag{8}$$

Equation (8) says that the correlation function vanishes identically if the distance between the two lattice edges under consideration is larger than two lattice spacing. This is a consequence of the simple form of the free energy given by Equ. (13) in I. As pointed out in [2], the absence of the dimer-dimer correlation beyond a certain distance, also found in the Sutherland-Rokhsar-Kivelson state of a quantum dimer model [4,5], is a property unique to the kagome lattice.

We can now compute all nonvanishing correlation functions by substituting (6) and (7) into (2), including correlations between two dimer edges within a unit cell and in two neighboring unit cells. There are 12 edges belonging to a unit cell as numbered in Fig. 1. Details of the computations, which are straightforward, will not be given. We tabulate the results in Tables 1-3.

3 Grassmannian approach

In a series of papers in 1980 [6] Samuel pointed out and explored the relation between the classical close-packed dimer problem and the free Grassmannian lattice field theory. He showed that the dimer generating function can be

represented as a fermionic path integral, and that dimer correlation functions are correlators occurring in a free fermion theory. He applied this formalism to dimers on the square lattice. Subsequently, Fendley *et al.* [7] extended the consideration to the triangular lattice using a slightly modified fermionic action; they also elucidated the field theory aspect of the formalism. Here we follow the formalism of [7] and apply it to the kagome lattice.

We begin with the Kasteleyn orientation of the lattice shown in Fig. 1 in I. Associate *real* Grassmannian variables η to lattice sites and consider the functional integral

$$Z = \int \exp\Big[(1/2)\sum_{i,j}\eta_i A_{ij}\eta_j\Big]\mathcal{D}\eta\,, \tag{9}$$

where A_{ij} is the matrix element of the $6MN \times 6MN$ Kasteleyn matrix A given by Equ. (1) in I and $\mathcal{D}\eta = \prod_i d\eta_i$.

It is well-known that the integral (9) is the Pfaffian of the matrix A, a fact which can be seen by expanding the exponential and using the result that the integral

$$\int \eta_{i_1}\ldots\eta_{i_{6MN}}\mathcal{D}\eta = \pm 1,$$

if $i_1\cdots i_{6MN}$ is a permutation of $1,\cdots,6MN$, and vanishes otherwise. Thus, the path integral (9) is identical to the dimer generating function.

For finite lattices with toroidal boundary conditions PBC considered in Section 2.1 in I, the dimer generating function is again expressed as a linear combination of four integrals corresponding to periodic boundary conditions with, or without, the reversal of boundary edge orientations as discussed in Section 2.1 in I. . In each case the exponent in (9) can be block-diagonalize by Fourier transforms of η_i.

For reverse edge orientations in both directions, for example, the Fourier transform is

$$\eta_{\alpha_p,\beta_q,a} = (MN)^{-1/2}\sum_{n,m}\eta_{n,m,a}e^{i\alpha_p n + i\beta_q m},$$

where (n,m) labels the position of unit cells, $a = 1,\ldots,6$ the sites within a unit cell, and $\alpha_p = (2p+1)\pi/N,\ p = 0,\ldots,N-1$, $\beta_q = (2q+1)\pi/M,\ q = 0,\ldots,M-1$.

After carrying out the Fourier transform, the functional integral (9) becomes

$$Z = \int \exp\Big\{\frac{1}{2}\sum_{p,q,a,b}\eta^*_{\alpha_p,\beta_q,a}\Big[A(\alpha_p,\beta_q)\Big]_{ab}\eta_{\alpha_p,\beta_q,b}\Big\}\mathcal{D}\eta, \tag{10}$$

where $A(\alpha,\beta)$ is the 6×6 matrix given in Equ. (7) in paper I and $\mathcal{D}\,\eta =$

$\prod_{p,q,a} d\eta_{\alpha_p,\beta_q,a}$. Further using relations

$$\eta^*_{\theta_p,\phi_q,a} = \eta_{\alpha_{N-1-p},\beta_{M-1-q},a},$$

and

$$\Big[A(\alpha_p,\beta_q)\Big]_{a,b} = \Big[-A(-\alpha_p,-\beta_q)\Big]_{ba} = \Big[-A(\alpha_{N-1-p},\beta_{M-1-q})\Big]_{ba}, \qquad (11)$$

we obtain from (10)

$$\begin{aligned} Z &= \int e^{(1/2)\sum_{p,q,a,b}\eta_{\alpha_{N+1-p},\beta_{M-1-q},a}\left[-A(\alpha_{N-1-p},\beta_{M-1-q})\right]_{ba}\eta^*_{\alpha_{N-1-p},\beta_{M-1-q},b}}\mathcal{D}\eta^* \\ &= \int e^{(1/2)\sum_{p,q,a,b}\eta^*_{\alpha_{N+1-p},\beta_{M-1-q},b}\left[A(\alpha_{N-1-p},\beta_{M-1-q})\right]_{ba}\eta_{\alpha_{N-1-p},\beta_{M-1-q},a}}\mathcal{D}\eta^* \\ &= \int e^{(1/2)\sum_{p,q,a,b}\eta^*_{\alpha_p,\beta_q,a}[A(\alpha_p,\beta_q)]_{ab}\eta_{\alpha_p,\beta_q,b}}\mathcal{D}\eta^* \end{aligned} \qquad (12)$$

where $\mathcal{D}\eta^* = \prod_{p,q,a} d\eta^*_{\alpha_p,\beta_q,a} = \prod_{p,q,a} d\eta_{\alpha_{N-1-p},\beta_{M-1-q},a} = \mathcal{D}\eta$. Here, we have used anticommutation relations of Grassmannian variables and renamed dummy variables p, q, a, b.

Next we use a well-known formula of Gaussian integrals of Grassmannian variables (see, e.g., (2.8) in the first reference in [6]) to write the product of (10) and (12) as

$$Z^2 = \prod_{p,q}\iint e^{\sum_{a,b}\eta^*_{\alpha_p,\beta_q,a}[A(\alpha_p,\beta_q)]_{ab}\eta_{\alpha_p,\beta_q,b}}\mathcal{D}\eta_{p,q}\mathcal{D}\eta^*_{p,q} = \prod_{p,q}\det|A(\alpha_p,\beta_q)|.$$

where $\mathcal{D}\eta_{p,q}\mathcal{D}\eta^*_{p,q} = \prod_a d\eta_{\alpha_p,\beta_q,a} d\eta^*_{\alpha_p,\beta_q,a}$. This is identically the expression $\det|A_4|$ given by Equ. (6) in I derived there using the method of Pfaffians.

The Grassmannian formalism is most useful in evaluating dimer-dimer correlation functions. To evaluate the correlation function (2), for example, we note that the dimer generating function can be written as

$$Z = \sum_{\text{covering}}\Big[\prod_{<ij>} w_{ij}^{n_{ij}}\Big],$$

where n_{ij} is defined in (1) and the summation is over all dimer coverings of the lattice. Thus, we have

$$\begin{aligned} \langle n_{ij}\rangle &= w_{ij}Z^{-1}\frac{\partial Z}{\partial w_{ij}} = w_{ij}Z^{-1}\int \eta_i\eta_j\exp(\mathcal{A})\mathcal{D}\eta, \\ \langle n_{ij}n_{k\ell}\rangle &= w_{ij}w_{k\ell}Z^{-1}\frac{\partial^2 Z}{\partial w_{k\ell}\partial w_{ij}} = w_{ij}w_{k\ell}Z^{-1}\int \eta_i\eta_j\eta_k\eta_\ell\exp(\mathcal{A})\mathcal{D}\eta. \end{aligned}$$

where $\mathcal{A} = (1/2)\sum_{i,j}\eta_i A_{ij}\eta_j$, and we have assumed that the Kasteleyn orientation is from i to j and from k to ℓ to get the correct sign for the Grass-

mannian integrals. These are precisely expressions of correlators occurring in a Gaussian theory and can be immediately written down. This gives

$$\begin{aligned}\langle n_{ij}\rangle &= w_{ij}A^{-1}_{ji}\\ \langle n_{ij}n_{k\ell}\rangle &= w_{ij}w_{k\ell}(A^{-1}_{ji}A^{-1}_{\ell k} - A^{-1}_{ki}A^{-1}_{\ell j} + A^{-1}_{\ell i}A^{-1}_{kj}),\end{aligned}$$

where A^{-1} is the inverse matrix of A with elements given by Equ. (1) in I. Here, we have used the Wick's theorem in expanding the four-point correlator.

Finally, the dimer correlation function (2) is given simply by the expression

$$c(ij;k\ell) = w_{ij}w_{k\ell}(-A^{-1}_{ki}A^{-1}_{\ell j} + A^{-1}_{\ell i}A^{-1}_{kj}), \tag{13}$$

where $A^{-1} \equiv G$ for a large lattice has been given in (6).

It can be verified that (13) gives rise to the same results as tabulated in Tables 1-3. Correlation functions of three or more dimers can be derived in a similar fashion. The advantage of using the Grassmannian method here is that it does not invoke products of large matrices needed in the Pfaffian approach.

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References

[1] F. Y. Wu and F. Wang, the preceeding paper.

[2] F. Wang and F. Y. Wu, *Phys. Rev. E*, **75** (2007), p.040105(R)

[3] M. E. Fisher and J. Stephenson, *Phys. Rev.*, **132** (1963), p.1411

[4] G. Misguich, D. Serban and V. Pasquier, *Phys. Rev. Lett.*, **89** (2002), p.137202

[5] D. S. Rokhsar, S. A. Kivelson, *Phys. Rev. Lett.*, **61** (1988), p.2376

[6] S. Samuel, *J. Math. Phys.*, **21** (1980), p.2806 ; *ibid.*, **21** (1980), p.2815 ; *ibid.*, **21** (1980), p.2820

[7] P. Fendley, R. Moessner and S. L. Sondhi, *Phys. Rev. B*, **66** (2002), p.214513

$(r_x, r_y) = (0,0)$												
	1	2	3	4	5	6	7	8	9	10	11	12
1	s	b	b	0	0	0	b	b	a	0	0	0
2	b	s	b	0	0	0	b	b	a	0	0	0
3	b	b	s	0	0	0	a	a	b	0	0	0
4	0	0	0	s	b	b	b	a	a	b	b	a
5	0	0	0	b	s	b	a	b	b	b	b	a
6	0	0	0	b	b	s	a	b	b	a	a	b
7	b	b	a	b	a	a	s	b	b	0	0	0
8	b	b	a	a	b	b	b	s	b	0	0	0
9	a	a	b	a	b	b	b	b	s	0	0	0
10	0	0	0	b	b	a	0	0	0	s	b	b
11	0	0	0	b	b	a	0	0	0	b	s	b
12	0	0	0	a	a	b	0	0	0	b	b	s

Table 1
Correlation function (2) between two dimers in the same unit cell, $a = 1/16, b = -1/16, s = 3/16$.

$(r_x, r_y) = (0, 1)$												
	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	b	a	a	0	0	0	0	0	0	0	0	0
11	a	b	b	0	0	0	0	0	0	0	0	0
12	a	b	b	0	0	0	0	0	0	0	0	0

Table 2
Correlation function (2) between two dimers in unit cells $(0, 0)$ and $(0, 1)$, $a = 1/16, b = -1/16$. Row indices $1, 2, \ldots, 12$ label edges in unit cell $(0, 0)$; column indices label edges in unit cell $(0, 1)$ (Cf. Fig. 1).

$(r_x, r_y) = (1, 0)$												
	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	0	0	0	0	b	a	b	0	0	0
2	0	0	0	0	0	0	a	b	a	0	0	0
3	0	0	0	0	0	0	b	a	b	0	0	0
4	0	0	0	0	0	0	0	0	0	b	a	b
5	0	0	0	0	0	0	0	0	0	a	b	a
6	0	0	0	0	0	0	0	0	0	b	a	b
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0

Table 3
Correlation function (2) between two dimers in unit cells $(0, 0)$ and $(1, 0)$, $a = 1/16, b = -1/16$. Row indices $1, 2, \ldots, 12)$ label edges in unit cell $(0, 0)$; column indices label edges in the unit cell $(1, 0)$ (Cf. Fig. 1).